… Sampling …
… Filtering …
… Reconstruction …

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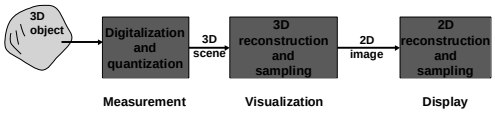
Overview

- Sampling of continuous signals
- Filtering of discrete signals
- Reconstruction of continuous signals
- Sampling aspects of volume rendering
- Voxelization of geometric objects

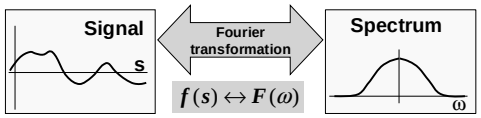
Why Sampling and Reconstruction?

- Real-world signals are continuous



- Computer representations are discrete

Signal and Spectrum



$$f(s) \leftrightarrow F(\omega)$$

- FT: decomposition of a signal into a sum of sinusoids, determined by frequency, amplitude and phase
- *f(x)* - representation in the spatial domain
- *F(ω)* - representation in the frequency domain

Fourier Transform

$$f(s) \leftrightarrow F(\omega)$$

- Direct FT: $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)\, e^{-i\omega s}\, ds$
- Inverse FT: $f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega s}\, d\omega$

Convolution

- Continuous case:

$$f(s) * h(s) = \int_{-\infty}^{\infty} f(s) \cdot h(s-\tau)\, d\tau$$

- Discrete case:

$$f[n] * h[n] = \sum_{k} f[k] \cdot h[n-k]$$

Convolution Theorem

- **Continuous case:**

$$FT\{f(t) * h(t)\} = F(\omega) \cdot H(\omega)$$

$$FT\{f(t) \cdot h(t)\} = F(\omega) * H(\omega)$$

- **Discrete case:**

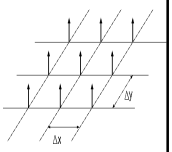
$$FT\{f[n] * h[n]\} = F(\Omega) \cdot H(\Omega)$$

$$FT\{f[n] \cdot h[n]\} = F(\Omega) * H(\Omega)$$

Sampling

- **Input signal:** $f_I(x, y)$
- **Ideal sampling function:**

$$s(x, y) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \delta(x - i\Delta_x, y - j\Delta_y)$$



- **Sampled signal:**

$$f_s(x, y) = f_I(x, y)s(x, y)$$

Sampling in the Frequency Domain

Spectrum of the sampled signal:

$$F_s(\omega_x, \omega_y) = \frac{1}{4\pi^2} F_I(\omega_x, \omega_y) * S(\omega_x, \omega_y)$$

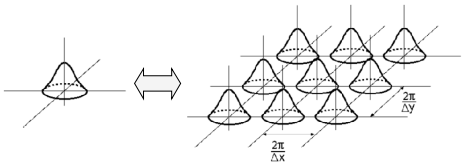
where

$$S(\omega_x, \omega_y) = \frac{4\pi^2}{\Delta_x \Delta_y} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \delta(\omega_x - i\omega_{xs}, \omega_y - i\omega_{ys})$$

and

$$\omega_{xs} = \frac{2\pi}{\Delta_x}, \omega_{ys} = \frac{2\pi}{\Delta_y}$$

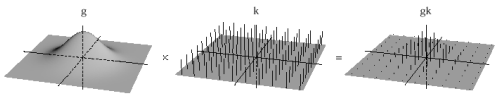
Sampling in the Frequency Domain



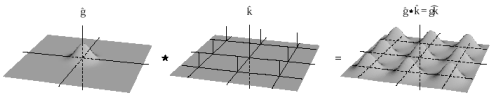
- **Sampling a signal causes replication of its spectrum in the frequency domain**
- **Nyquist criterion:**

$$\omega_s > 2\omega_m$$

Sampling of a Signal



In the spatial domain



In the frequency domain

Real-world Sampling Process

- **The sampling function has a volume**
- **It is approximated by a Gaussian:**

$$s_r(x, y, z) = \exp\left(-\left(\frac{x^2}{2\sigma_x} + \frac{y^2}{2\sigma_y} + \frac{z^2}{2\sigma_z}\right)\right)$$

- **A Gaussian is a low-pass filter, i.e. it causes blurred image, blunt edges**

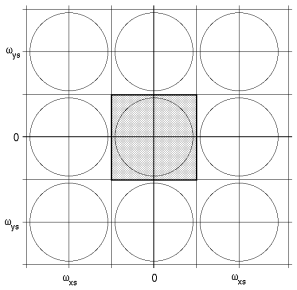
Reconstruction

Direct reconstruction of a continuous signal from discrete samples:

- **The spectrum replicas have to be suppressed by a reconstruction filter**
- **The ideal reconstruction filter is a box function:**

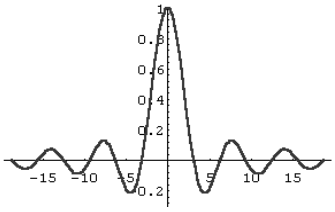
$$R(\omega_x, \omega_y) = \begin{cases} K \text{ if } |\omega_x| \le \omega_{xc} \wedge |\omega_y| \le \omega_{yc} \\ 0 \text{ else} \end{cases}$$

Reconstruction



Ideal Reconstruction Filter

$$\text{box}\,(\omega) \leftrightarrow \text{sinc}\,(s) = \frac{\sin(s)}{s}$$



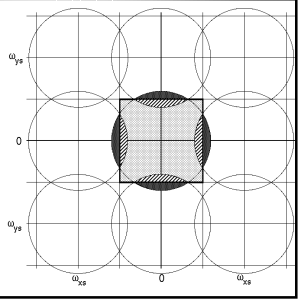
Ideal Reconstruction Filter

$$\text{box}(\omega) \leftrightarrow \text{sinc}(s) = \frac{\sin(s)}{s}$$

- **Bounded support in the frequency domain, but unbounded in the spatial domain**
- **Can't be realized practically**
- **It has to be approximated (bounded):**
  - **The approximations have unbounded support in the frequency domain**

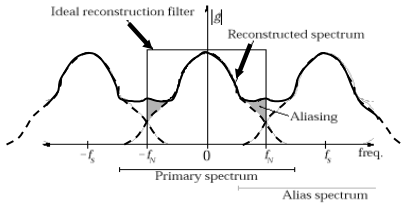
Problems with the Reconstruction (aliasing)

- **If the Nyquist criterion is not fulfilled**
- **If the reconstruction filter is too big or too small**



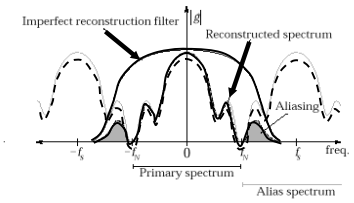
Prealiasing

- **Wrong frequencies appear if the Nyquist criterion is not fulfilled:**



Postaliasing

- Wrong frequencies appear also if the reconstruction filter support is too broad



Classification of 3D reconstruction filters

- Separable:

$$h(x,y,z) = h_s(x)\ h_s(y)\ h_s(z)$$

  - Sequential application along the axes
  - Computational complexity: ***3n***
- Spherically symmetrical:
  - Computational complexity: ***n³***

Separable Filters

- Order 0: nearest neighbor

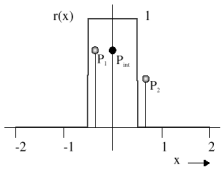
$$h_s = 1 \text{ if } |x| < 0.5$$

- Order 1: linear interpolation

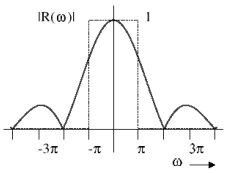
$$h_s = 1 - |x| \text{ if } |x| < 1$$

- Order 3: cubic filters:
  - Cubic B-spline
  - Catmull-Rom spline

Nearest Neighbor

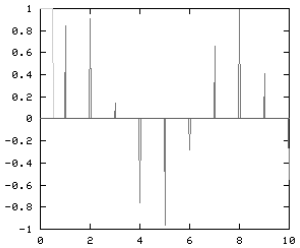


Filter

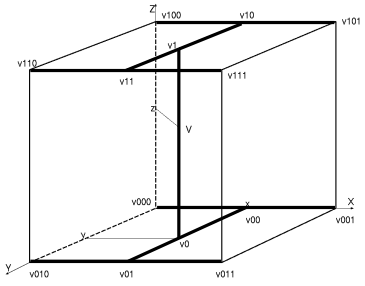


Spectrum

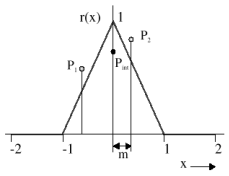
Nearest Neighbor Filtering



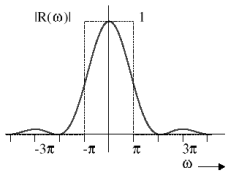
Trilinear Reconstruction Filter



Linear Interpolation

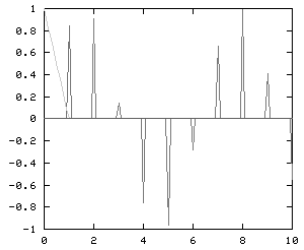


Filter

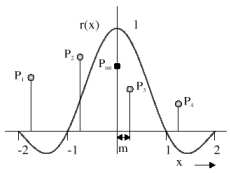


Spectrum

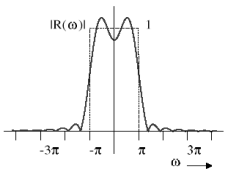
Linear Filtering



Truncated *sinc* Filter

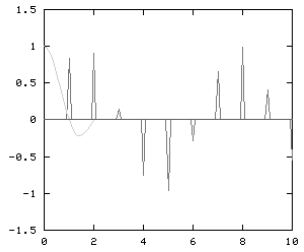


Filter



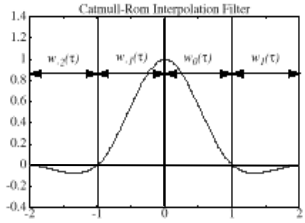
Spektrum

Truncated *sinc* Filtering

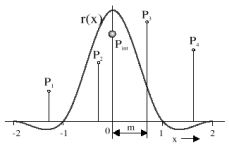


Catmull-Rom Spline

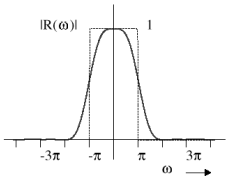
- **Piecewise cubic, C1-smooth**



Catmul-Rom Spline

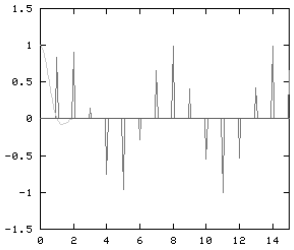


Filter



Spectrum

Catmull-Rom Spline Filtering



Other Separable Filters

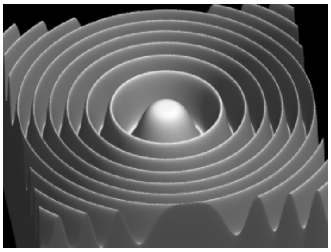
- **Gaussian filter**

$$h_s(x) = \exp(-\frac{x^2}{2\sigma^2}), \quad |x| < x_m$$

- **Windowed *sinc* filter**

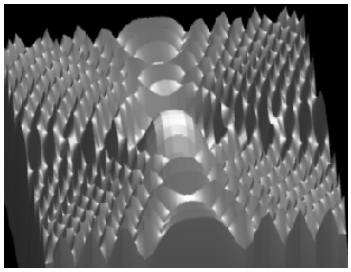
$$h_s(x) = \left(1 + \cos(\frac{\pi x}{x_m})\right) \mathrm{sinc}(\frac{4x}{x_m}), \quad |x| < x_m$$

An Example

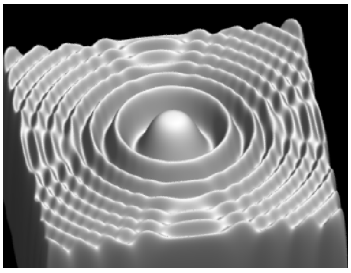


F_N = 10, sampling 40 x 40 x 40

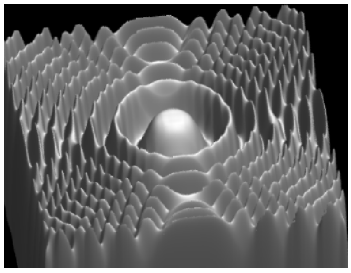
Reconstruction by the Trilinear Filter



Reconstruction by the Cubic B-spline Filter



Reconstruction by the Catmull-Rom Spline Filter



Reconstruction by the Windowed *sinc* Filter